

A New Fluid Dynamical Model Coupling Heat with Application to Interior Separations

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Abstract

Based on the Boussinesq equations and the equation of state, a new fluid dynamical model coupling heat is established. Furthermore, the conditions for interior separation, which are determined by the initial conditions, are obtained. This result is derived from the new model by using the interior separation theorem which was established by T. Ma and S. Wang in [10, 12]. The most important application of this result is to predict when and where tornado and hurricane will occur.

keywords

The Boussinesq equations, The equation of state, Interior separation, Tornado, Hurricane.

1 Introduction

Separations of fluid flows are fundamental issues in fluid dynamics. The physical and numerical descriptions of boundary layer separation go back to the pioneer work of Prandtl [12] in 1904. Boundary layer separation is the phenomenon that a vortex is generated from the boundary. It is a very common phenomenon in geophysical dynamics, such as gyres of gulf stream,

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separation of atmospheric circulation near mountain. There are many researches [2, 4–6, 8, 11, 12, 16, 18] on boundary layer separation over the past one hundred years. It is noticed that the geometric theory of 2D incompressible flows was initiated by the authors in [5, 11, 12] to study the structural stability and transition of 2D incompressible fluid flows. The results give new rigorous characterization of boundary layer separation (see Ref. [11] and references therein). Recently, a predicable condition for boundary layer separation of 2D incompressible fluid flows, which is determined by initial values and external forces, was obtained in [8, 18].

For the interior separation, the fluid flows can separate from the interior, generating a vortex. The kinematic theory for interior separation of 2D incompressible flows was initiated by T. Ma and S. Wang in [10, 12]. The interior separation problems for the fluid dynamical equations were discussed in [14, 15]. It is well known that interior separation phenomena correspond to tornado and hurricane in geophysical fluid dynamics and climate dynamics, which are caused by the horizontal thermal motion.

We know that the classical Boussinesq equations [3] are invalid to investigate interior separation because of no horizontal thermal expanding forces, which are given as follows

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u} - \frac{1}{\rho} \nabla p + (1 - \alpha T) g \vec{k} + F, \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \Delta T + Q, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \quad (1.1)$$

where ρ is the density of the fluid, g is acceleration due to gravity, α is the coefficient of thermal expansion of the fluid, $\vec{k} = (0, 0, 1)$ is the unit vector in the x_3 -direct, μ is the dynamic viscosity coefficient, κ is the thermal diffusion coefficient, Q is the thermal source and F is the external force. The unknown functions are the velocity field $\mathbf{u} = (u_1, u_2, u_3)$, the pressure function p and the temperature function T .

In order to study the interior separation problems by using the fluid dynamical equations coupling horizontal heat, we have to modify the Boussinesq equations (1.1). For this purpose, we set up a new fluid dynamical model for the fluid flows, which is given as follows

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u} + \nu \nabla \operatorname{div} \mathbf{u} - \beta T \nabla \varphi - \delta \nabla \varphi - \beta \nabla T + \mathbf{F}, \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \Delta T + Q, \\ \frac{\partial \varphi}{\partial t} + (\mathbf{u} \cdot \nabla) \varphi + \operatorname{div} \mathbf{u} = 0, \end{cases} \quad (1.2)$$

where β and δ are constants, μ and ν are the dynamic viscosity coefficients, $\varphi = \ln \rho$ is the density function and \mathbf{F} is the external force.

Applying the kinematic theory of interior separation, which is established by T. Ma and S. Wang, into the model (1.2), we develop a theory to predict when, where and how the interior separation occurs, and apply the theory to study the tornado and hurricane problems.

The paper is organized as follows. In Section 2, we recall some preliminaries, including definition of topologically equivalent, and definitions of boundary layer separation and interior separation, and the interior separation theorem, and the structural stability theorem. Section 3 establishes a new fluid dynamical model coupling heat for fluid flows. In Section 4, we derive interior separation theorem for the new model and give the predictable conditions for tornado and hurricane. Finally, we give some physical interpretations to predict when and where the interior separation will occur.

2 Preliminaries

Let $M \subset \mathbb{R}^2$ be a C^r ($r \geq 1$) closed domain with boundary, $C^r(M, \mathbb{R}^2)$ be the space of all C^r vector fields on M . Let

$$D^r(M, \mathbb{R}^2) = \{\mathbf{w} \in C^r(M, \mathbb{R}^2) | \text{div} \mathbf{w} = 0\}. \quad (2.1)$$

Definition 2.1. [12, 15]. Two vector fields \mathbf{u} and $\mathbf{v} \in D^r(M, \mathbb{R}^2)$ are called topologically equivalent if there exists a homeomorphism of $\Upsilon : M \rightarrow M$, which takes the orbits of \mathbf{u} to orbits of \mathbf{v} and preserves their orientation.

Definition 2.2. [10]. A vector field $\mathbf{v} \in D^r(M, \mathbb{R}^2)$ is called structurally stable in $D^r(M, \mathbb{R}^2)$ if there exists a neighborhood $\mathcal{O} \subset D^r(M, \mathbb{R}^2)$ of \mathbf{v} such that for any $\mathbf{u} \in \mathcal{O}$, \mathbf{u} and \mathbf{v} are topologically equivalent.

Definition 2.3. [11, 12, 15]. We call that boundary layer separation of a 2D vector field \mathbf{u} occurs at t_0 , if $\mathbf{u}(x, t)$ is topologically equivalent to the structure of Figure 1(a) for any $t < t_0$, and to the structure of Figure 1(b) for $t > t_0$. That is, if $t < t_0$, $\mathbf{u}(x, t)$ is topologically equivalent to a parallel flow, and if $t > t_0$, $\mathbf{u}(x, t)$ separates a vortex. Furthermore, we call that boundary layer separation occurs at \bar{x} , if \bar{x} is an isolated boundary singular point at time $t = t_0$.



Figure 1: Definition of boundary layer separation

Definition 2.4. [10, 12, 15]. We call that interior separation of a 2D vector field \mathbf{u} occurs at t_0 , if $\mathbf{u}(x, t)$ is topologically equivalent to the structure of Figure 2(a) for any $t < t_0$, and to the structure of Figure 2(b) for $t = t_0$, and to the structure of Figure 2(c) or Figure 2(c') for $t > t_0$. That is, if $t < t_0$, the flow given by Figure 2(a) exhibits no singular point in the neighborhood of x_0 . At $t = t_0$, $\mathbf{u}(x, t_0)$ is given by Figure 2(b), which has an isolated singular point $x_0 \in M$ with index zero. When $t > t_0$, the flow pattern is given by Figure 2(c) or Figure 2(c') in the back flow region.

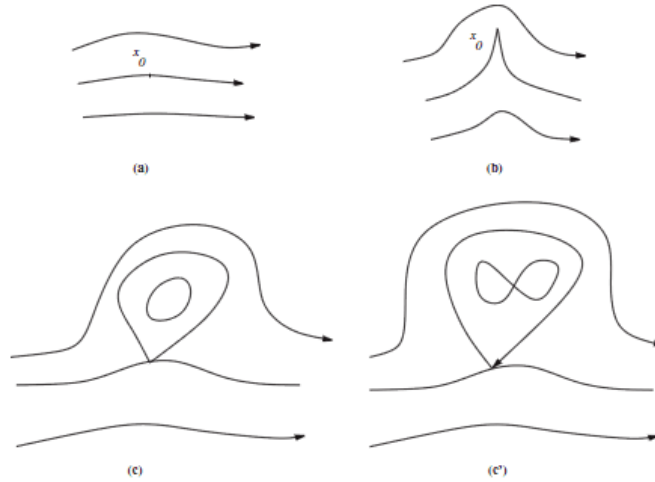


Figure 2: Definition of interior separation

Let $\mathbf{u} \in C^1([0, \tau], D^r(M, \mathbb{R}^2))$ ($\tau > 0$) be a one-parameter family of divergence-free vector fields and its Taylor expansion at $t = t_0$ ($0 < t_0 < \tau$)

is written as

$$\begin{cases} \mathbf{u}(x, t) = \mathbf{u}^0(x) + \mathbf{u}^1(x)(t - t_0) + o(|t - t_0|), \\ \mathbf{u}^0(x) = \mathbf{u}(x, t_0), \quad x \in M, \\ \mathbf{u}^1(x) = \frac{\partial \mathbf{u}}{\partial t} \big|_{t=t_0}. \end{cases} \quad (2.2)$$

For convenience, we denote $D\mathbf{u}^0(x_0)$ is the Jacobian matrix of $\mathbf{u}^0(x)$ at x_0 , which can be expressed as

$$D\mathbf{u}^0(x_0) = \begin{pmatrix} \frac{\partial u_1^0(x_0)}{\partial x_1} & \frac{\partial u_1^0(x_0)}{\partial x_2} \\ \frac{\partial u_2^0(x_0)}{\partial x_1} & \frac{\partial u_2^0(x_0)}{\partial x_2} \end{pmatrix}, \quad (2.3)$$

where $\mathbf{u}^0(x_0)$ is defined in (2.2).

The following assumption of the vector field \mathbf{u} which is crucial in the interior separation theorem.

Assumption 2.5. *Let $x_0 \in \mathring{M}$ be an isolated degenerated singular point of $\mathbf{u}^0(x)$. Suppose that*

$$\text{ind}(\mathbf{u}^0, x_0) = 0, \quad (2.4)$$

$$D\mathbf{u}^0(x_0) \neq 0, \quad (2.5)$$

$$\mathbf{u}^1(x_0) \cdot e_2 \neq 0, \quad (2.6)$$

where $D\mathbf{u}^0(x_0)$ is the Jacobian matrix of $\mathbf{u}^0(x)$ at x_0 , $\text{ind}(\mathbf{u}^0, x_0)$ is the index of \mathbf{u}^0 at the isolated singular point x_0 and e_2 is as defined in the following (2.8).

Let $x_0 \in \mathring{M}$ be an isolated degenerated singular point of $\mathbf{u}^0(x)$ with nonzero Jacobian: $D\mathbf{u}^0(x_0) \neq 0$. Since $D\mathbf{u}^0(x_0)$ is a degenerated nonzero matrix, $D\mathbf{u}^0(x_0)$ has an eigenvector e_1 satisfying

$$D\mathbf{u}^0(x_0)e_1 = 0, \quad |e_1| = 1. \quad (2.7)$$

Let e_2 be a unit vector, orthogonal to e_1 , and satisfies that

$$D\mathbf{u}^0(x_0)e_2 = \alpha e_1, \quad (2.8)$$

for some constant $\alpha \neq 0$.

We now recall interior separation theorem and interior structure bifurcation theorem established by T. Ma and S. Wang [10, 12] in the following.

Theorem 2.6. *Let $\mathbf{u} \in C^1([0, \tau], D^r(M, \mathbb{R}^2))$ ($\tau > 0, r \geq 1$) satisfy the Assumption 2.5. Then the vector field \mathbf{u} has the interior separation at $x_0 \in \mathring{M}$ as shown schematically in Figure 2(c) or Figure 2(c').*

The following theorem provides necessary and sufficient conditions for structural stability of a divergence-free vector field in $D^r(M, \mathbb{R}^2)$.

Theorem 2.7. *[9, 10, 12]. A divergence-free vector field $\mathbf{u} \in D^r(M, \mathbb{R}^2)$ is structurally stable in $D^r(M, \mathbb{R}^2)$ if and only if*

1. *u is regular;*
2. *all interior saddle points of u are self-connected; and*
3. *each saddle point of u on ∂M is connected only to saddle points on the same connected component of ∂M .*

Moreover, the set of all structurally stable vector fields is open and dense in $D^r(M, \mathbb{R}^2)$.

Remark 2.8. *When $M \subset \mathbb{R}^2$ is without boundary, the results similar to Theorem 2.6 hold true as well.*

3 A new fluid dynamical model coupling heat

Based on the Newton's second law, the motion of fluid flows can be governed by the Navier-Stokes equations coupling heat expressed as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u} + \nu \nabla \operatorname{div} \mathbf{u} - \frac{1}{\rho} \nabla p + (1 - \alpha T) g \vec{k} + \mathbf{F}, \quad (3.1)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the pressure, g is the acceleration due to gravity, α is the coefficient of thermal expansion of the fluid, $\vec{k} = (0, 0, 1)$ is the unit vector in the x_3 -direct, ρ is the mass density, μ, ν are the dynamic viscosity constants and \mathbf{F} is the external force.

The conservation of mass takes the following form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0, \quad (3.2)$$

which, with the constant density, is reduced to $\operatorname{div} \mathbf{u} = 0$.

It is well known that different fluid flows correspond to different expressions for equation of state. When it comes to gas, the fluid is approximated as the ideal gas, which is enough in gas dynamical applications in [1] and the equation of state is given by

$$p\mathcal{V} = \mathcal{N}\mathcal{R}T, \quad (3.3)$$

where \mathcal{N} is the number of moles of gas, \mathcal{V} is the volume of the system, $\mathcal{R} = 8314 \text{ J}/(\text{kg} \cdot \text{mol} \cdot \text{K})$ is the gas constant, p is the pressure and T is the temperature.

It is easy to see that (3.3) can be rewritten as follows

$$\begin{aligned}
p &= \frac{\mathcal{N} \mathcal{R} T}{\mathcal{V}} \\
&= \frac{\mathcal{R}}{M} \cdot \frac{\mathcal{N} M}{\mathcal{V}} T \\
&= R \rho T,
\end{aligned} \tag{3.4}$$

where M is the molecular weight, $R = \frac{\mathcal{R}}{M}$ is specific gas constant depending on the type of the gas and ρ is the density.

When the fluid flow is liquid, the relationship between the pressure p and the temperature T can be approximated by

$$p = \rho(\sigma T + \gamma), \tag{3.5}$$

where σ, γ are constants.

Combining (3.4) and (3.5), we conclude that

$$p = \rho(\beta T + \delta), \tag{3.6}$$

where β and δ are depending on the types of fluid flows, i.e.

$$\beta = \begin{cases} R, & \text{for gas,} \\ \sigma, & \text{for liquid,} \end{cases}$$

and

$$\delta = \begin{cases} 0, & \text{for gas,} \\ \gamma, & \text{for liquid.} \end{cases}$$

Let the density ρ of the fluid flows be

$$\rho(x, t) = e^{\varphi(x, t)}, \tag{3.7}$$

where $x \in \mathbb{R}^n (n = 2, 3)$, $t \in [0, \infty)$. Note that $\varphi = \ln \rho$. Moreover, $-\frac{1}{\rho} \nabla p$ in (3.1) can be rewritten as

$$-\frac{1}{\rho} \nabla p = -\frac{1}{\rho} \nabla(\rho(\beta T + \delta)) = -\beta T \nabla \varphi - \beta \nabla T - \delta \nabla \varphi. \tag{3.8}$$

For simplicity, we denote $\mathbf{F} = F + (1 - \alpha T)g\vec{k}$, (3.1) and (3.8) imply that

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u} + \nu \nabla \operatorname{div} \mathbf{u} - (\beta T + \delta) \nabla \varphi - \beta \nabla T + \mathbf{F}. \tag{3.9}$$

From the Heat Conduction law, we obtain that

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \Delta T + Q, \tag{3.10}$$

where κ is thermal diffusion coefficient and Q is the thermal source.

Together (3.2) with (3.7), we also get

$$\frac{\partial \varphi}{\partial t} + (\mathbf{u} \cdot \nabla) \varphi + \operatorname{div} \mathbf{u} = 0. \quad (3.11)$$

From (3.9) – (3.11), we immediately drive the model (1.2).

Remark 3.1. *The model (1.2) can govern the motion of 2D or 3D fluid flows for both compressible and incompressible. It is obvious that the model (1.2) contains the horizontal thermal expanding force $-\beta T \nabla \varphi - \beta \nabla T$.*

Remark 3.2. *Instead of the previous pressure term p in (1.1), our model (1.2) only relates to temperature T , which is crucial to study the natural phenomena (tornado, hurricane) caused by the horizontal thermal expanding forces.*

4 Main Results

We firstly study the interior separation for 2D incompressible fluid flows with the horizontal thermal motion.

4.1 Interior separation theorem for 2D incompressible fluid flows coupling heat

In large scale, the density ρ of fluid flows can be approximatively seen as a constant. In this case, φ is also a constant from (3.7). Then, the equations (1.2) are rewritten as

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u} - \beta \nabla T + \mathbf{F}, \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \Delta T + Q, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \quad (4.1)$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ is the velocity field, T is the temperature, μ is the dynamic viscosity coefficient, κ is the thermal diffusion coefficient, Q is the thermal source and \mathbf{F} is the external force with the initial condition

$$\mathbf{F} |_{t=0} = F^0(x). \quad (4.2)$$

The corresponding initial conditions of (4.1) are given by

$$\begin{cases} \mathbf{u} |_{t=0} = \Psi(x), \\ T |_{t=0} = T^0(x). \end{cases} \quad (4.3)$$

A natural method for studying interior separation of such a family $\mathbf{u}(\cdot, t)$ and $\mathbf{T}(\cdot, t)$ are to Taylor expand them near initial time $t = 0$, and then to analyze their structure in a neighborhood of $t = 0$, which is given by

$$\begin{cases} \mathbf{u}(x, t) = \Psi(x) + t\mathbf{u}^1(x) + o(|t|), \\ T(x, t) = T^0(x) + tT^1(x) + o(|t|), \\ \Psi(x) = \mathbf{u}(x, 0), \quad \mathbf{u}^1(x) = \frac{\partial \mathbf{u}}{\partial t} \big|_{t=0}, \\ T^0(x) = T(x, 0), \quad T^1(x) = \frac{\partial T}{\partial t} \big|_{t=0}. \end{cases} \quad (4.4)$$

From (4.1)-(4.4), we deduce that

$$\mathbf{u}^1(x) = \frac{\partial \mathbf{u}}{\partial t} \big|_{t=0} = \mu \Delta \Psi - (\Psi \cdot \nabla) \Psi - \beta \nabla T^0 + F^0, \quad (4.5)$$

which obviously implies that

$$\mathbf{u}(x, t) = \Psi(x) + t[\mu \Delta \Psi - (\Psi \cdot \nabla) \Psi - \beta \nabla T^0 + F^0] + o(|t|). \quad (4.6)$$

Let

$$\mathbf{v}(x, t) = \Psi(x) + t[\mu \Delta \Psi - (\Psi \cdot \nabla) \Psi - \beta \nabla T^0 + F^0]. \quad (4.7)$$

It is obvious that

$$\mathbf{u}(x, t) = \mathbf{v}(x, t) + o(|t|) \quad \text{for } 0 \leq t \ll 1.$$

Now, we make the following assumption.

Assumption 4.1. *Let the functions Ψ, T^0, F^0 , given by (4.2) and (4.3), be the solutions of the following equations*

$$\operatorname{div} \Psi(x) = 0, \quad \text{for } \forall x \in \mathbb{R}^2, \quad (4.8)$$

$$\frac{\partial \Psi_1}{\partial x_2} \frac{\partial \Psi_2}{\partial x_1} + \left(\frac{\partial \Psi_1}{\partial x_1} \right)^2 + \frac{\beta}{2} \Delta T^0 - \frac{1}{2} \operatorname{div} F^0 = 0, \quad \text{for } \forall x \in \mathbb{R}^2, \quad (4.9)$$

and they satisfy the conditions that there exists an $\bar{x} \in U \subset \mathbb{R}^2$ such that \bar{x} is an isolated singular point of $\mathbf{v}(x, t_0)$ with the Jacobian matrix $D\mathbf{v}(\bar{x}, t_0) \neq 0$, and there exists a time t_0 with $0 \leq t_0 \ll 1$ such that the vector $\mathbf{v}(x, t) \neq 0$ for $x \in U$, $0 \leq t < t_0$, and

$$[\mu \Delta \Psi - (\Psi \cdot \nabla) \Psi - \beta \nabla T^0 + F^0] \big|_{x=\bar{x}} \cdot e_2 \neq 0, \quad (4.10)$$

where $\mathbf{v}(x, t)$ is as given by (4.7) and e_2 is as defined in (2.8).

The interior separation theorem of the model (4.1) is given by the following theorem.

Theorem 4.2. *Let $\mathbf{u}(x, t), T(x, t)$ be the solution of equations (4.1), (4.3). If the functions Ψ, T^0, F^0 satisfy Assumption 4.1, then the vector field \mathbf{u} has an interior separation near (\bar{x}, t_0) .*

Proof. First, we show that the vector field $\mathbf{v}(x, t)$ has an interior separation near (\bar{x}, t_0) . From (4.7) – (4.9), we get

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \operatorname{div}\{\Psi(x) + t[\mu\Delta\Psi - (\Psi \cdot \nabla)\Psi - \beta\nabla T^0 + F^0]\} \\ &= -t\{\operatorname{div}(\Psi \cdot \nabla)\Psi + \beta\Delta T^0 - \operatorname{div} F^0\} \\ &= -t\{2\frac{\partial\Psi_1}{\partial x_2}\frac{\partial\Psi_2}{\partial x_1} + 2(\frac{\partial\Psi_1}{\partial x_1})^2 + \beta\Delta T^0 - \operatorname{div} F^0\} \\ &= 0. \end{aligned} \tag{4.11}$$

By Theorem 2.6, we only need to prove that the divergence-free $\mathbf{v}(x, t)$ satisfies the following. From Assumption 4.1, we know that U is the neighborhood of the singular point \bar{x} and

$$\mathbf{v}(x, t) \neq 0 \quad \text{for} \quad x \in U, \quad 0 \leq t < t_0,$$

which implies

$$\operatorname{ind}(\mathbf{v}(x, t_0), \bar{x}) = 0, \quad \det D\mathbf{v}(\bar{x}, t_0) = 0. \tag{4.12}$$

Moreover, it is easy to see that the Jacobian matrix $D\mathbf{v}(\bar{x}, t_0) \neq 0$ and

$$\frac{\partial \mathbf{v}}{\partial t} \big|_{x=\bar{x}, t=t_0} \cdot e_2 = [\mu\Delta\Psi - (\Psi \cdot \nabla)\Psi - \beta\nabla T^0 + F^0] \big|_{x=\bar{x}} \cdot e_2 \neq 0. \tag{4.13}$$

From the above discussion, we see that the Assumption 2.5 is satisfied. So, we can obtain that the vector \mathbf{v} has an interior separation near (\bar{x}, t_0) .

Second, we will prove that the velocity \mathbf{u} has an interior separation near (\bar{x}, t_0) . By (4.6) and (4.7), we see that

$$\mathbf{u}(x, t) = \mathbf{v}(x, t) + o(|t|). \tag{4.14}$$

Diving by t on the both sides of (4.14), we obtain

$$\frac{1}{t}\mathbf{u}(x, t) = \frac{1}{t}\mathbf{v}(x, t) + O(|t|), \quad (O(|t|) \rightarrow 0 \quad \text{as} \quad t \rightarrow 0). \tag{4.15}$$

It is easy to see that $\exists \tau$ with $0 < t < \tau \ll 1$ such that for $t \in (0, \tau)$

$$\frac{1}{t}\mathbf{u} \quad \text{is a perturbation of} \quad \frac{1}{t}\mathbf{v}. \tag{4.16}$$

Due to the Theorem 2.7 and Remark 2.8, we know that $\frac{1}{t}\mathbf{u}$ and $\frac{1}{t}\mathbf{v}$ are topologically equivalent for $t_0 < t < \tau$.

On the other hand, $\frac{1}{t}\mathbf{v}$ has the same orbit structure as \mathbf{v} . It is noticed that \mathbf{v} has an interior separation near (\bar{x}, t_0) in the first discussion. Hence, $\frac{1}{t}\mathbf{u}$ has the interior separation near (\bar{x}, t_0) , which implies \mathbf{u} has the interior separation near (\bar{x}, t_0) . □

Remark 4.3. *It is easy to see that (4.8) and (4.9) guarantee $\text{div} \mathbf{v} = 0$. By the topology degree theory, the vector field $\mathbf{v}(x, t)$ has no zero point in U for $0 \leq t < t_0$ implies that $\text{ind}(\mathbf{v}(x, t_0), \bar{x}) = 0$.*

Remark 4.4. *The conditions given in Assumption 4.1 are determined by the initial external force (4.2) and the initial conditions (4.3). It is easy to see that \mathbf{v} has an interior separation near (\bar{x}, t_0) is enough to judge the interior separation of \mathbf{u} . This theorem provides an approach to predict the interior separation depending on the initial functions. More specially, this result provides an approach to predict the tornado or the hurricane just by initial velocity, temperature and external force under the Assumption 4.1.*

4.2 The predictable conditions for the interior separation

The main objective of this subsection is to give a specific application of the Theorem 4.2, which are depending on the initial functions. We consider the model (4.1) with initial conditions given by

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u} - \beta \nabla T + \mathbf{F}, \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \Delta T + Q, \\ \text{div} \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \Psi(x), \quad T|_{t=0} = T^0(x), \end{cases} \quad (4.17)$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ is the velocity field, T is the temperature function, μ is the dynamic viscosity coefficient, κ is the thermal diffusion coefficient, Q is the thermal source and \mathbf{F} is the external force.

To make the equations nondimensional, let

$$\begin{aligned} (x, t) &= (Lx', \frac{L^2}{\mu}t'), \\ (\mathbf{u}, T) &= (\frac{\mu}{L}\mathbf{u}', \theta T'), \\ \mathbf{F} &= \frac{\mu^2}{L^3}\mathbf{F}', Q = \frac{\mu\theta}{L^2}Q', \\ \Psi &= \frac{\mu}{L}\Psi', T^0 = \theta T'^0. \end{aligned}$$

Here μ is the dynamic viscosity coefficient, L is the diameter of the neighborhood of singular point \bar{x} and θ is the unit of temperature.

Omitting the primes, the model (4.17) can be rewritten as

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \Delta \mathbf{u} - \frac{L^2 \beta \theta}{\mu^2} \nabla T + \mathbf{F}, \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \frac{\kappa}{\mu} \Delta T + Q, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \Psi(x), \quad T|_{t=0} = T^0(x). \end{cases} \quad (4.18)$$

The following theorem is the application of the Theorem 4.2 under nondimensional initial functions.

Theorem 4.5. *Let $\mathbf{u}(x, t)$, $T(x, t)$ be the solution of equations (4.18). If the initial conditions $\Psi(x), T^0$ satisfy*

$$\begin{cases} \Psi = (\Psi_1, \Psi_2) = (0, 1 + C_1 x_1^2), \\ T^0 = C_2 + C_3 x_2, \end{cases} \quad (4.19)$$

and the initial external force satisfies

$$\mathbf{F}^0 = (C_4 x_2, 0), \quad (4.20)$$

where $C_i (i = 1, 2, 3, 4)$ are nonzero constants. Then vector field \mathbf{u} exists an interior separation near $(0, t_0)$ ($t_0 = \frac{1}{\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1}$ with $\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1 \gg 1$) and μ is diffusion coefficient.

Proof. From (4.19) and (4.20), it is obviously that the functions $\Psi(x), T^0, \mathbf{F}^0$ are the solutions of (4.8) and (4.9). By (4.7), (4.19) and (4.20), we know that

$$\begin{cases} \mathbf{v}_1(x, t_0) = C_4 x_2 t_0 = 0, \\ \mathbf{v}_2(x, t_0) = 1 + t_0 [2C_1 - \frac{L^2 \beta \theta}{\mu^2} C_3] - C_1 x_1^2 t_0 = 0. \end{cases} \quad (4.21)$$

When $t = t_0 = \frac{1}{\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1}$ in (4.21), we immediately derive that $(0, 0)$ is the isolated singular point of $\mathbf{v}(x, t_0)$.

Combining (4.7) and $\Psi(\bar{x}) = \Psi(0) \neq 0$, we see that there exists an open set $U \subset \mathbb{R}^2$ and a time t_0 with $0 \leq t_0 \ll 1$ such that the vector $\mathbf{v}(x, t) \neq 0$ for $x \in U$, $0 \leq t < t_0$.

Next, we will prove the Jacobian matrix $D\mathbf{v}(0, t_0)$ of $\mathbf{v}(0, t_0)$ is nonzero degenerated matrix. By the simple calculation from (4.21), we get

$$D\mathbf{v}(0, t_0) = \begin{pmatrix} 0 & C_4 t_0 \\ 0 & 0 \end{pmatrix}. \quad (4.22)$$

It is obvious that $\det D\mathbf{v}(0, t_0) = 0$. Hence, $(0, 0)$ is an isolated degenerated singular point of $\mathbf{v}(x, t_0)$ with $D\mathbf{v}(0, t_0) \neq 0$.

It is clear that

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} \big|_{x=\bar{x}=0, t=t_0} \cdot \mathbf{e}_2 &= [\mu \Delta \Psi - (\Psi \cdot \nabla) \Psi - \frac{L^2 \beta \theta}{\mu^2} \nabla T^0 + F^0] \big|_{x=\bar{x}=0} \cdot \mathbf{e}_2 \\
&= (0, 2C_1 - \frac{L^2 \beta \theta}{\mu^2} C_3) \cdot (0, 1) \\
&= 2C_1 - \frac{L^2 \beta \theta}{\mu^2} C_3 \neq 0.
\end{aligned} \tag{4.23}$$

Therefore, the Assumption 4.1 is satisfied, the conclusion follows from the Theorem 4.2. \square

4.3 Physical Interpretations

In the following, we give the physical interpretation of the tornado and hurricane related to Theorem 4.6.

Physical Interpretation 4.7. It is easy to see that the tornado and hurricane occurs near $(0, t_0)$ ($t_0 = \frac{1}{\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1}$ is small enough) under the assumptions (4.19) and (4.20). It is also noticed that $t_0 = \frac{1}{\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1}$ (C_1 and C_3 are nonzero constant) with $\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1 \gg 1$ is the time at which tornado and hurricane occur. The singular point \bar{x} , which can be obtained by (4.21), is the place where the tornado and hurricane occur.

Physical Interpretation 4.8. In reality, the diameter L of the neighborhood of the tornado's and hurricane's center \bar{x} is very large. Although the dimensionless time $t_0 = \frac{1}{\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1}$ (C_1 and C_3 are nonzero constant) with $\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1 \gg 1$ is small, we can easy to see that the real predictable time $\bar{t} = \frac{L^2}{\mu} t_0$ is not small.

Physical Interpretation 4.9. From $\frac{L^2 \beta \theta}{\mu^2} C_3 - 2C_1 \gg 1$, we deduce that

$$\frac{L^2 \beta \theta}{\mu^2} C_3 \gg 1 \tag{4.24}$$

or

$$-2C_1 \gg 1. \tag{4.25}$$

It is noticed that (4.24) implies that large difference in temperature can cause tornado and hurricane. It is also noticed that (4.25) means that the large initial velocity can cause tornado and hurricane. More importantly, the meaning of (4.24) and (4.25) accord with the physical fact.

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